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Oscillator quantum groups from *R*-matrix method

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Abstract. The R-matrix method is applied to the one-dimensional oscillator group. The resulting quantum groups and dual algebras are characterized. It is shown that both bosonic and fermionic dual algebras occur and that true deformations of the bosonic and fermionic oscillator algebras are obtained.

Résumé. La méthode de la matrice R est appliquée au groupe de l'oscillateur à une dimension. Les groupes quantiques et algèbres duales associées sont alors caractérisés. Nous montrons qu'il apparaît des algèbres duales de type bosonique et fermionique. De plus, nous obtenons de vraies déformations des algèbres de l'oscillateur bosonique et fermionique.

1. Introduction

The R-matrix method [1–5] is used to generate quadratic algebraic relations between the basic elements of a given group G, which are compatible with the coproduct as derived from the group structure. We recall briefly how to apply this method. Let T be a faithful representation of G, then we get the desired quadratic relations in writing

$$RT_1T_2 = T_2T_1R \tag{1.1}$$

where as usual $T_1 = T \otimes I$, $T_2 = I \otimes T$.

The external consistency of the algebra thus defined is ensured if R satisfies the well known quantum Yang–Baxter equation (QYBE).

This program has been extensively applied to semi-simple group where the QYBE is a necessary condition. In particular for $gl(2)$, all the R-matrices verifying QYBE have been given by Hietarinta [6]. Two classes of R-matrices appear: the first contains matrices continuously related to the identity matrix and the second matrices continuously related to a diagonal matrix with at least a negative element. The interesting object is actually the dual algebra which plays the same role with respect to the quantum group as the Lie algebra for the group. In the situation referred to above, the corresponding dual algebras can be termed bosonic and fermionic quantum algebras.

Majid [7] indicated that the R-matrix method could be applied to non-semisimple groups and, in particular, to the one-dimensional Heisenberg and oscillator groups. In a previous paper [8], we thoroughly considered the case of the Heisenberg group. It was shown that in this case we have to replace QYBE by a weakened version, insofar as a matrix commuting with $T \otimes T \otimes T$ was not necessarily a multiple of the identity, but no fermionic structure appeared.

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In the present paper, we extend the application to the one-dimensional oscillator group. Rather surprisingly, we get a large number of matrices, between which we distinguish two families continuously related to diagonal matrices, the identity matrix and one with a negative element. Thus by duality we recover in analogy with the $gl(2)$ case [9, 10] algebras which can be called bosonic and fermionic oscillator quantum algebras. Besides, as in the case of the Heisenberg group [8], we have to verify a weakened version of QYBE for the same reason, but in contrast, it happens that for each concerned deformed algebra, we can find a R-matrix verifying the strict QYBE.

Our paper is organized as follows. in section 2, we discuss the properties of the R-matrix and we give the algebraic relations defining the various types of quantum groups as well in the bosonic as in the fermionic case. In section 3, we derive the bosonic dual algebras and in section 4, the dual fermionic algebras which are indeed new algebraic deformations of $gl(1/1)$ compared with the results of Burdik *et al* [10]. In section 5, we discuss these results from an algebraic point of view; it is shown that by a convenient change of generators, some of these algebras are actually isomorphic to the usual oscillator algebra, but even in this case, the deformation can be read on the coproduct, so that the deformation process is meaningful in the category of bialgebras. Appendices A, B, C and D contain many technical devices needed for performing the various computations.

2. Bosonic and fermionic oscillator quantum groups

Let us start with a three-dimensional faithful matrix representation of the oscillator group

$$T = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \eta & \gamma \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

where α, β, η, γ are generators of an algebra A, provided with a structure of Hopf algebra where the comultiplication implied by the oscillator group law is given by

$$\begin{aligned} \Delta\alpha &= 1 \otimes \alpha + \alpha \otimes \eta \\ \Delta\gamma &= \eta \otimes \gamma + \gamma \otimes 1 \\ \Delta\beta &= 1 \otimes \beta + \beta \otimes 1 + \alpha \otimes \gamma \\ \Delta\eta &= \eta \otimes \eta. \end{aligned} \tag{2.2}$$

To construct the quantum oscillator group we use systematically the R-matrix approach [1–5]. In that case the R-matrix is a 9×9 matrix $\{r_{ij},\ i, j = 1, \ldots, 9\}$ defined by equation (1.1).

It is clear that, in particular, if R is the identity matrix, the equation (1.1) gives the commutative bialgebra for $\alpha, \beta, \eta, \gamma$. But there exists another diagonal R-matrix compatible with the coproduct (2.2) for which we get a non-commutative (non-deformed) bialgebra. Indeed, we have $R = \mathrm{diag}(1, 1, 1, 1, -1, 1, 1, 1, 1)$ and the bialgebra is defined by

$$\begin{aligned} &\{\alpha, \eta\} = 0 \qquad \{\gamma, \eta\} = 0 \qquad \alpha^2 = 0 \qquad \gamma^2 = 0 \\ &[\alpha, \beta] = 0 \qquad [\beta, \gamma] = 0 \qquad [\alpha, \gamma] = 0 \qquad [\beta, \eta] = 0 \end{aligned} \tag{2.3}$$

where { , } denotes the anticommutator.

So here the R-matrix approach allows us to deal with deformations of two types of bialgebras. We want to identify them with bosonic and fermionic quantum oscillator groups. Indeed, it will be proved in what follows that their dual algebras are deformations of the usual bosonic and fermionic oscillator algebras, respectively.

Starting from this point of view and setting to zero every linear relation between the generators, we have to impose on the R-matrix some limiting conditions which finally result in the following set of quadratic relations:

$$\begin{aligned}
&r_{33}(\alpha\beta-\beta\alpha)=r_{56}\alpha^2-r_{15}\eta\gamma-(r_{13}-r_{26}-r_{46})\alpha+r_{16}(1-\eta)\\
&r_{33}(\beta\gamma-\gamma\beta)=-r_{25}\gamma^2+r_{59}\eta\alpha-(r_{26}+r_{28}-r_{39})\gamma-r_{29}(1-\eta)\\
&r_{33}(\alpha\gamma-\gamma\alpha)=-r_{25}\eta\gamma+r_{56}\eta\alpha\\
&r_{33}(\beta\eta-\eta\beta)=r_{45}\eta\gamma-r_{56}\alpha\eta\\
&r_{33}\eta\alpha+(r_{11}-r_{33}-r_{55})\alpha\eta=r_{45}(\eta-\eta^2)\\
&r_{33}\eta\gamma+(r_{11}-r_{33}-r_{55})\gamma\eta=r_{58}(\eta-\eta^2)\\
&(r_{11}-r_{55})\alpha^2=r_{15}(1-\eta^2)+(r_{25}+r_{45})\alpha\\
&(r_{11}-r_{55})\gamma^2=r_{59}(1-\eta^2)+(r_{56}+r_{58})\gamma.
\end{aligned}\tag{2.4}$$

To ensure the consistency of this set of equations, we get the relation

$$(r_{11}-r_{55})(r_{11}-2r_{33}-r_{55})=0.\tag{2.5}$$

In the particular case where $r_{33}=0$, we have no relation between the generators insofar as we add the conditions

$$\begin{aligned}
&r_{15}=r_{16}=r_{18}=r_{25}=r_{29}=r_{45}=r_{49}=r_{56}=r_{58}=r_{59}=0\\
&r_{39}=r_{26}+r_{28}\qquad r_{46}=r_{13}-r_{26}.
\end{aligned}\tag{2.6}$$

The family of corresponding R-matrices is denoted by R_0 and said to belong to the zero type.

If now we assume $r_{33}\neq 0$, we have to distinguish two cases which we consider separately in what follows.

2.1. The bosonic case: $r_{55}=r_{11}$

Again requiring internal consistency, we must have

$$r_{18}=-r_{16}\qquad r_{45}=-r_{25}\qquad r_{49}=-r_{29}\qquad r_{58}=-r_{56}\qquad r_{15}=r_{59}=0\tag{2.7}$$

and, furthermore, Jacobi's identity gives

$$r_{25}r_{56}=0\qquad r_{25}(r_{13}-r_{26}-r_{46})=0\qquad r_{56}(r_{26}+r_{28}-r_{39})=0.\tag{2.8}$$

So two distinct types of bosonic oscillator quantum groups appear:

$$\begin{aligned}
&(1^\circ)\ \text{Type I:}\quad r_{25}=0\qquad r_{39}=r_{26}+r_{28}.\\
&(2^\circ)\ \text{Type II:}\quad r_{25}=0\qquad r_{56}=0.
\end{aligned}\tag{2.9}$$

The case $r_{56}=0$, $r_{46}=r_{13}-r_{26}$ is identical to the type I after the change of generators $\alpha\leftrightarrow\gamma$. Taking $r_{33}=1$ and setting

$$\begin{aligned}
&p=-r_{13}+r_{26}+r_{46}\qquad q=r_{26}+r_{28}-r_{39}\\
&a=r_{56}\qquad b=r_{16}\qquad c=r_{29}
\end{aligned}\tag{2.10}$$

the commutation relations are

$$\begin{aligned}
&[\alpha,\beta]=a\alpha^2+p\alpha+b(1-\eta)\qquad [\alpha,\eta]=0\\
&[\alpha,\gamma]=a\alpha\eta\qquad [\beta,\eta]=-a\alpha\eta\\
&[\gamma,\eta]=-a(\eta^2-\eta)\qquad [\beta,\gamma]=-q\gamma-c(1-\eta).
\end{aligned}\tag{2.11}$$

Type I corresponds to $q = 0$, whilst type II corresponds to $a = 0$.

The corresponding matrix R can be written as

$$
\begin{aligned}
R = \big\{ & \{r_{11}, 0, r_{13}, 0, 0, b, r_{17}, -b, r_{19}\}, \\
& \{0, 1, 0, r_{11} - 1, 0, r_{26}, 0, r_{28}, c\}, \\
& \{0, 0, 1, 0, 0, 0, r_{11} - 1, 0, -q + r_{26} + r_{28}\}, \\
& \{0, r_{11} - 1, 0, 1, 0, p + r_{13} - r_{26}, 0, -p + r_{17} - r_{28}, -c\}, \\
& \{0, 0, 0, 0, r_{11}, a, 0, -a, 0\}, \\
& \{0, 0, 0, 0, 0, 1, 0, r_{11} - 1, 0\}, \\
& \{0, 0, r_{11} - 1, 0, 0, 0, 1, 0, q + r_{13} + r_{17} - r_{26} - r_{28}\}, \\
& \{0, 0, 0, 0, 0, r_{11} - 1, 0, 1, 0\}, \\
& \{0, 0, 0, 0, 0, 0, 0, 0, r_{11}\}\big\}.
\end{aligned}
\tag{2.12}
$$

2.2. *The fermionic case:* $r_{55} = r_{11} - 2r_{33}$

Here internal consistency gives

$$r_{16} = r_{18} = r_{25} = r_{29} = r_{45} = r_{49} = r_{56} = r_{58} = 0 \tag{2.13}$$

and

$$r_{15}(r_{13} - r_{26} - r_{46}) = 0 \qquad r_{59}(r_{26} + r_{28} - r_{39}) = 0. \tag{2.14}$$

As in the previous case, $r_{33} = 1$ and we take, together with (2.10),

$$z = r_{15} \qquad x = r_{59} \tag{2.15}$$

so that the constraint (2.14) is written

$$pz = 0 \qquad qx = 0. \tag{2.16}$$

As before, due to the invariance by the change $\alpha \leftrightarrow \gamma$, we derive only three types of fermionic oscillator quantum groups:

$$
\begin{aligned}
&(1^\circ)\ \text{Type I:} \quad p = q = 0 \\
&(2^\circ)\ \text{Type II:} \quad q = z = 0 \\
&(3^\circ)\ \text{Type III:} \quad x = z = 0.
\end{aligned}
\tag{2.17}
$$

The algebraic relations are now

$$
\begin{aligned}
&\{\alpha, \eta\} = 0 \qquad \{\gamma, \eta\} = 0 \qquad \alpha^2 = \tfrac{1}{2} z(1 - \eta^2) \qquad \gamma^2 = \tfrac{1}{2} x(1 - \eta^2) \\
&[\alpha, \beta] = z\gamma\eta + p\alpha \qquad [\alpha, \gamma] = 0 \\
&[\gamma, \beta] = x\alpha\eta + q\gamma \qquad [\beta, \eta] = 0.
\end{aligned}
\tag{2.18}
$$

The corresponding R-matrix can be written as

$$
\begin{aligned}
R = \big\{ \quad & \{r_{11}, 0, r_{13}, 0, z, 0, r_{17}, 0, r_{19}\}, \\
& \{0, 1, 0, r_{11} - 1, 0, r_{26}, 0, r_{28}, 0\}, \\
& \{0, 0, 1, 0, 0, 0, r_{11} - 1, 0, -q + r_{26} + r_{28}\}, \\
& \{0, r_{11} - 1, 0, 1, 0, p + r_{13} - r_{26}, 0, -p + r_{17} - r_{28}, 0\}, \\
& \{0, 0, 0, 0, r_{11} - 2, 0, 0, 0, x\}, \\
& \{0, 0, 0, 0, 0, 1, 0, r_{11} - 1, 0\}, \\
& \{0, 0, r_{11} - 1, 0, 0, 0, 1, 0, q + r_{13} + r_{17} - r_{26} - r_{28}\}, \\
& \{0, 0, 0, 0, 0, r_{11} - 1, 0, 1, 0\}, \\
& \{0, 0, 0, 0, 0, 0, 0, 0, r_{11}\}\big\}.
\end{aligned}
\tag{2.19}
$$

We see that in each case the R-matrices contain more parameters than those appearing in the structure relations (2.11) and (2.17). Moreover $r_{11} \neq 0$ as R is non-singular.

As we already noted in [8], internal consistency is equivalent to a weak version of QYBE in the following sense:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}W$$

where R_{12}, R_{13}, R_{23} have the usual meaning and W is a matrix that commutes with $T \otimes T \otimes T$, which is not necessarily the identity matrix.

In analogy with what we did in [8], a generic matrix R may be given. Such a matrix is labelled by $y = r_{11} - 1$, the set (p, q, a, b, c) in case A and the set (p, q, x, z) in case B, characterizing the type of concerned quantum group. The other irrelevant parameters (not involved in (2.11) and (2.17)) being taken equal to zero. The generic R can be built in the following way: if R_1, R_2 belong to the same type under the condition $y_1 + y_2 \neq 0$, then R is given equal to $R_1 S^{-1} R_2$ where S is a convenient matrix in the set R_0. This generic R does not generally verify the QYBE, since the consistency of (2.11) and (2.17) does not force W to be the identity. However, in the present case, it can be shown that for fixed (p, q, a, b, c) or (p, q, x, z) there exists a non-generic matrix R satisfying QYBE where obviously the irrelevant parameters are not necessarily zero.

3. Deformed bosonic oscillator algebras

In this section we construct the dual algebras of the type I and type II bosonic quantum groups given by the structure relations (2.11). As we know, these dual algebras are related to the quantum groups as the universal envelopping algebras are related to the Lie groups. In this way, we get various deformations of the usual oscillator algebra. In all cases, the quantum oscillator group has the basis

$$\beta^k \alpha^\ell \eta^m \gamma^n \qquad k, \ell, m, n \in \mathbb{N}. \tag{3.1}$$

We define the generators of the dual algebra by

$$
\begin{aligned}
(A, \beta^k \alpha^\ell \eta^m \gamma^n) &= \delta_{k0}\delta_{\ell 1}\delta_{n0} \\
(B, \beta^k \alpha^\ell \eta^m \gamma^n) &= \delta_{k1}\delta_{\ell 0}\delta_{n0} \\
(C, \beta^k \alpha^\ell \eta^m \gamma^n) &= \delta_{k0}\delta_{\ell 0}\delta_{n1} \\
(H, \beta^k \alpha^\ell \eta^m \gamma^n) &= m\delta_{k0}\delta_{\ell 0}\delta_{n0}.
\end{aligned}
\tag{3.2}
$$

If P_1, P_2 are any two elements in this dual algebra, their product is defined by

$$(P_1 P_2, \beta^k \alpha^\ell \eta^m \gamma^n) = (P_1 \otimes P_2, \Delta\beta^k \alpha^\ell \eta^m \gamma^n) \tag{3.3}$$

where Δ means that we take the coproduct of subsequent factors. In applying this formula to the generators defined in (3.2), we have to treat the two types of quantum oscillator groups separately.

(1°) Type I

We set

$$\Delta\beta^k = \Gamma^k_{r,s,t;r',s',t',u'} \beta^r \alpha^s \eta^t \otimes \beta^{r'} \alpha^{s'} \eta^{t'} \gamma^{u'}$$

$$\Delta\gamma^n = \Delta^n_{uv,v'} \eta^u \gamma^v \otimes \gamma^{v'}$$

with summation over repeated indices. Taking into account

$$\Delta\alpha^\ell = \binom{\ell}{\ell'} \alpha^{\ell'} \otimes \alpha^{\ell-\ell'} \eta^{\ell'}$$

$$\Delta\eta^m = \eta^m \otimes \eta^m$$

and systematically using the commutation relations (2.11), we get

$$\Delta\beta^k \alpha^\ell \eta^m \gamma^n = \Gamma^k_{r,s,t;r',s',t',u'} \binom{\ell}{\ell'} \Delta^n_{uv,v'} \beta^r \alpha^{s+\ell'} \eta^{t+m+u} \gamma^v$$
$$\otimes \beta^{r'} \alpha^{s'+\ell-\ell'} \eta^{t'} Q^{u',\ell-\ell',m+\ell'}_{u''}(\eta) \gamma^{u''+v'} \tag{3.4}$$

where the polynomials $Q^{r,s,s'}_{r'}(\eta)$ are defined by

$$(\gamma - as\eta)^r \eta^{s'} = Q^{r,s,s'}_{r'}(\eta) \gamma^{r'}. \tag{3.5}$$

For any regular function $f(\eta)$ we have

$$\left(\begin{matrix} A \\ B \\ C \end{matrix}, \beta^k \alpha^\ell f(\eta) \gamma^n\right) = \left(\begin{matrix} A \\ B \\ C \end{matrix}, \beta^k \alpha^\ell \gamma^n\right) f(1)$$
$$(H, \beta^k \alpha^\ell f(\eta) \gamma^n) = \delta_{k0} \delta_{\ell 0} \delta_{\eta 0} \frac{\mathrm{d}f}{\mathrm{d}\eta}(1). \tag{3.6}$$

Using this formula systematically we are able to express the various commutators as summations over the coefficients $\Gamma^k_{r,s,t;r',s',t',u'}$, $\Delta^n_{uv,v'}$, which can be evaluated from recursion formulae over them. The interested reader will find some indication as to how to proceed in appendices A, B and C.

We finally get

$$[A, B] = [B, C] = [B, H] = 0$$
$$[A, C] = F'(B)$$
$$[C, H] = bF(B) + \frac{\mathrm{e}^{-aC} - 1}{a} \tag{3.7}$$
$$[A, H] = A\big(1 - abF(B)\big) - aHF'(B) + \frac{c}{ab}\big(F''(B) - \mathrm{e}^{\rho B}\big).$$

Here the function $F(x)$ is given by

$$F(x) = \frac{1}{\rho_1\rho_2}\left(1 + \frac{\rho_2 e^{\rho_1 x} - \rho_1 e^{\rho_2 x}}{\rho_1 - \rho_2}\right) \tag{3.8}$$

ρ_1, ρ_2 being the roots of the second-order equation

$$\rho^2 - p\rho + ab = 0$$

i.e.

$$\rho_1 = \tfrac{1}{2}\big(p + \sqrt{p^2 - 4ab}\big) \qquad \rho_2 = \tfrac{1}{2}\big(p - \sqrt{p^2 - 4ab}\big). \tag{3.9}$$

Let us remark that we have the following limits when ab goes to zero:

$$\begin{aligned}
&\lim_{ab\to 0} F(x) = \frac{1}{p^2}(e^{px} - 1) - \frac{x}{p} \\
&\lim_{ab\to 0} F'(x) = \frac{e^{px} - 1}{p} \\
&\lim_{ab\to 0} \frac{1}{ab}\big(F''(x) - e^{px}\big) = \frac{e^{px} - 1}{p^2} - \frac{x}{p} e^{px}.
\end{aligned} \tag{3.10}$$

This algebra will be discussed in more detail in the conclusion.

(2°) Type II

Since only two commutators are different from zero, we get simpler formulae; indeed we have

$$\Delta\gamma^n = \binom{n}{n'} \eta^{m'} \gamma^{n-n'} \otimes \gamma^{n'}$$

and the recurrence formula for $\Gamma^k_{r,s,t;r',s',t',u'}$ is easier to manage as it can be seen in appendix A. Thus we start with

$$\Delta\beta^k\alpha^\ell\eta^m\gamma^n = \Gamma^k_{r,s,t;r',s',t',u'}\binom{\ell}{\ell'}\binom{n}{n'}\beta^r\alpha^{s+\ell'}\eta^{t+m+n'}\gamma^{n-n'} \otimes \beta^{r'}\alpha^{s'+\ell-\ell'}\eta^{t'+\ell'+m}\gamma^{u'+n'}$$

and we obtain the same formal expressions as before for the various commutators in performing the limit at $a = 0$ provided we note that

$$\lim_{a\to 0}(-a)^{u'} = \delta_{u'0}. \tag{3.11}$$

The evaluation of the various concerned coefficients can now be done easily (cf appendix B). Finally we get

$$\begin{aligned}
&[A, B] = [B, C] = [B, H] = 0 \\
&[A, C] = \frac{e^{(p+q)B} - 1}{p+q} \\
&[C, H] = -C - b\frac{e^{qB}}{p+q}\left(\frac{e^{pB} - 1}{p} + \frac{e^{-qB} - 1}{q}\right) \\
&[A, H] = A - c\frac{e^{pB}}{p+q}\left(\frac{e^{qB} - 1}{q} + \frac{e^{-pB} - 1}{p}\right).
\end{aligned} \tag{3.12}$$

This algebra is isomorphic to the usual oscillator algebra if we introduce

$$
\begin{aligned}
B' &= \frac{e^{(p+q)B}-1}{p+q} \\
C' &= C + b\frac{e^{qB}}{p+q}\left(\frac{e^{pB}-1}{p} + \frac{e^{-qB}-1}{q}\right) \\
A' &= A - c\frac{e^{pB}}{p+q}\left(\frac{e^{qB}-1}{q} + \frac{e^{-pB}-1}{p}\right)
\end{aligned}
\tag{3.13}
$$

instead of B, C, A. Therefore, at this stage, all the deformation parameters disappear. But, if we consider the dual algebra as a bialgebra, these parameters are playing a defining role in the coproduct of the generators; indeed we have

$$
\begin{aligned}
\Delta A &= I \otimes A + A \otimes e^{pB} \\
\Delta B &= I \otimes B + B \otimes I \\
\Delta C &= I \otimes C + C \otimes e^{qB} \\
\Delta H &= I \otimes H + H \otimes I + bA \otimes \frac{e^{pB}-1}{p} - cC \otimes \frac{e^{qB}-1}{q}
\end{aligned}
\tag{3.14}
$$

so that the deformation process takes place in the category of Hopf algebras.

4. Deformed fermionic oscillator algebras

Since in the structure relations (2.17) defining the fermionic quantum groups α^2 and γ^2 are always given in terms of η, a complete basis of the fermionic quantum groups is given by

$$\beta^k \eta^\ell \alpha^i \gamma^j \qquad k, \ell \in \mathbb{N} \quad i, j = 0, 1. \tag{4.1}$$

We define generators A, B, C, H in the dual bialgebra by the relations

$$
\begin{aligned}
(A, \beta^k \eta^\ell \alpha^i \gamma^j) &= \delta_{k0}\delta_{i1}\delta_{j0} \\
(B, \beta^k \eta^\ell \alpha^i \gamma^j) &= \delta_{k1}\delta_{i0}\delta_{j0} \\
(C, \beta^k \eta^\ell \alpha^i \gamma^i) &= \delta_{k0}\delta_{i0}\delta_{j1} \\
(H, \beta^k \eta^\ell \alpha^i \gamma^j) &= \ell\delta_{k0}\delta_{i0}\delta_{j0}.
\end{aligned}
\tag{4.2}
$$

As usual, the algebraic product of two elements in the dual algebra is given by

$$(XY, \beta^k \eta^\ell \alpha^i \gamma^j) = (X \otimes Y, \Delta\beta^k \eta^\ell \alpha^i \gamma^j). \tag{4.3}$$

We consider the various cases separately.

(1°) Type I

We prove recurrently that $\Delta\beta^k$ has the following form:

$$\Delta\beta^k = \Gamma^{k,ij}_{rs;r's'} \beta^r \eta^s \alpha^i \gamma^j \otimes \beta^{r'} \eta^{s'} \alpha^i \gamma^j + \Delta^{k,ij}_{rs;r's'} \beta^r \eta^s \alpha^i \gamma^j \otimes \beta^{r'} \eta^{s'} \alpha^{i+1} \gamma^{j+1} \tag{4.4}$$

where summation is over repeated indices and $i+1$, $j+1$ are taken mod 2. Moreover from the relation

$$\Delta\beta^{k+1} = \Delta\beta^k \Delta\beta$$

we get recurrence relation between the various coefficients useful for the computations to follow (see appendix D).

Applying equation (4.3), we obtain the values of $[A, B]$, $[B, C]$, $[B, H]$, $[A, H]$, $[C, H]$, $\{A, C\}$, A^2 and C^2 on the basis (4.1) in terms of summations over the coefficients $\Gamma^{k,ij}_{rs;r's'}$, $\Delta^{k,ij}_{rs;r's'}$. These summations can be evaluated by using the recurrence relations of appendix D, where we give some meaningful examples for interested reader. With similar methods, we evaluate B^n, B^nA, B^nC on the basis (4.1). Collecting together all these results and quoting only the non-zero quantities, we have

$$\{A, C\} = \frac{\sinh 2B\sqrt{xz}}{2\sqrt{xz}} \qquad A^2 = \frac{1-\cosh 2B\sqrt{xz}}{4z} \qquad C^2 = \frac{1-\cosh 2B\sqrt{xz}}{4x}$$

$$[A, H] = \frac{1}{2}A(1+\cosh 2B\sqrt{xz}) + x\frac{\sinh 2B\sqrt{xz}}{2\sqrt{xz}}C \tag{4.5}$$

$$[C, H] = -\frac{1}{2}C(1+\cosh 2B\sqrt{xz}) - z\frac{\sinh 2B\sqrt{xz}}{2\sqrt{xz}}A.$$

When x and z go to zero, we get the algebraic relations of the usual one-dimensional fermionic algebra.

Let us now build the coproduct. For any element X of the dual algebra, it is defined by

$$(\Delta X, \beta^k\eta^\ell\alpha^i\gamma^j \otimes \beta^{k'}\eta^{\ell'}\alpha^{i'}\gamma^{j'}) = (X, \beta^k\eta^\ell\alpha^i\gamma^j\beta^{k'}\eta^{\ell'}\alpha^{i'}\gamma^{j'}).$$

In re-ordering the product on the right-hand side, we use the following recursively proved relations:

$$\alpha^i\gamma^j\beta^k = \frac{1}{2}\left[\left(\beta+\eta(i+j)\sqrt{xz}\right)^k + \left(\beta-\eta(i+j)\sqrt{xz}\right)^k\right]\alpha^i\gamma^j$$
$$-\frac{1}{2}\left(\frac{z}{x}\right)^{(i-j)/2}\left[\left(\beta+\eta(i+j)\sqrt{xz}\right)^k - \left(\beta-\eta(i+j)\sqrt{xz}\right)^k\right]\alpha^{i+1}\gamma^{j+1} \tag{4.6}$$

where $i+j$, $i+1$, $j+1$ are taken mod 2.

Noting that

$$\left(B^n(-1)^H, \beta^k\eta^\ell\alpha^i\gamma^j\right) = \delta_{i0}\delta_{j0}\delta_{kn}(-1)^\ell n!$$

$$\left(B^n(-1)^H A, \beta^k\eta^\ell\alpha^i\gamma^j\right) = \delta_{i1}\delta_{j0}\delta_{kn}(-1)^\ell n!$$

$$\left(B^n(-1)^H C, \beta^k\eta^\ell\alpha^i\gamma^j\right) = \delta_{i0}\delta_{j1}\delta_{kn}(-1)^\ell n!$$

we finally get

$$\Delta A = 1\otimes A + A\otimes(-1)^H\cosh B\sqrt{xz} - xC\otimes(-1)^H\frac{\sinh B\sqrt{xz}}{\sqrt{xz}}$$

$$\Delta B = 1\otimes B + B\otimes 1$$

$$\Delta C = 1\otimes C + C\otimes(-1)^H\cosh B\sqrt{xz} - zA\otimes(-1)^H\frac{\sinh B\sqrt{xz}}{\sqrt{xz}} \tag{4.7}$$

$$\Delta H = 1\otimes H + H\otimes 1 - zA\otimes(-1)^H(\cosh B\sqrt{xz})A$$
$$-\sqrt{xz}A\otimes(-1)^H(\sinh B\sqrt{xz})C + \sqrt{xz}C\otimes(-1)^H(\sinh B\sqrt{xz})A$$
$$+xC\otimes(-1)^H(\cosh B\sqrt{xz})C.$$

(2°) Type II

In this case, we prove the following form of $\Delta\beta^k$:

$$\Delta\beta^k = \Gamma^{0,k}_{rs;r's'}\beta^r\eta^s \otimes \beta^{r'}\eta^{s'} + \Gamma^{1,k}_{rs;r's'}\beta^r\eta^s \otimes \beta^{r'}\eta^{s'}\alpha + \Gamma^{2,k}_{rs;r's'}\beta^r\eta^s\alpha \otimes \beta^{r'}\eta^{s'}\gamma.$$

The various commutators and anticommutators are again expressed on the basis (4.1) by summations on the $\Gamma^{\nu,k}_{rs;r's'}$, $\nu = 0, 1, 2$, which can be evaluated from the recurrence relations over them much more easily than before.

We finally get

$$\begin{aligned}
&\{A, C\} = \frac{e^{pB}-1}{p} \qquad A^2 = -\frac{x}{2}\left(\frac{e^{pB}-1}{p}\right)^2 \qquad C^2 = 0 \\
&[A, H] = A - x\frac{e^{pB}-1}{p}C \qquad [C, H] = -C \\
&[A, B] = [B, C] = [B, H] = 0.
\end{aligned} \tag{4.8}$$

The coproduct is easily obtained by using the formulae

$$\begin{aligned}
&\alpha\beta^k = (\beta+p)^k\alpha \qquad \alpha\gamma\beta^k = (\beta+p)^k\alpha\gamma \\
&\gamma\beta^k = \beta^k\gamma - \frac{x}{p}\big((\beta+p)^k - \beta^k\big)\eta\alpha
\end{aligned} \tag{4.9}$$

and we find

$$\begin{aligned}
&\Delta A = 1\otimes A + A\otimes e^{pB}(-1)^H + \frac{x}{p}C\otimes(1-e^{pB})(-1)^H \\
&\Delta B = 1\otimes B + B\otimes 1 \\
&\Delta C = 1\otimes C + C\otimes(-1)^H \\
&\Delta H = 1\otimes H + H\otimes 1 - xC\otimes(-1)^H C.
\end{aligned} \tag{4.10}$$

(3°) Type III

The form of $\Delta\beta^k$ is even simpler than previously. Indeed we have

$$\Delta\beta^k = \Gamma^{0,k}_{rs;r's'}\beta^r\eta^s \otimes \beta^{r'}\eta^{s'} + \Gamma^{1,k}_{rs;r's'},\beta^r\eta^s\alpha \otimes \beta^{r'}\eta^{s'}\gamma.$$

As the evaluations are very easy to do, we give the final results for both algebraic relations and coproduct directly. We have

$$\begin{aligned}
&\{A, C\} = \frac{e^{(p+q)B}-1}{p+q} \qquad A^2 = C^2 = 0 \\
&[A, H] = A \qquad [C, H] = -C \\
&[A, B] = [B, C] = [B, H] = 0
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&\Delta A = A\otimes e^{pB}(-1)^H + 1\otimes A \\
&\Delta B = 1\otimes B + B\otimes 1 \\
&\Delta C = C\otimes e^{qB}(-1)^H + 1\otimes C \\
&\Delta H = 1\otimes H + H\otimes 1.
\end{aligned} \tag{4.12}$$

5. Conclusion

5.1. The bosonic case

To get a deeper view of the quantum algebra associated with the type I bosonic quantum group, let us consider its possible irreducible representations. As B commutes with all other generators, we can expect it to be a multiple of the identity $B = \omega I$.

Now, we have two cases to discuss.

(1) $1-ab\,F(\omega) \neq 0$. In this case $1-abF(B)$ is an invertible operator in the representation and we can introduce

$$\begin{cases} A' = A - aH\dfrac{F'(B)}{1-abF(B)} + \dfrac{c}{ab}\dfrac{F''(B) - \mathrm{e}^{pB}}{1-abF(B)} \\ C' = \dfrac{\mathrm{e}^{aC}-1}{a} - b\dfrac{F(B)}{1-abF(B)} \\ H' = \dfrac{H}{1-abF(B)}. \end{cases} \tag{5.1}$$

With this change of generators and according to (3.7) we get the commutation relations

$$\begin{cases} [A', C'] = \dfrac{F'(B)}{1-abF(B)} = \dfrac{F'(\omega)}{1-abF(\omega)}I \\ [A', H'] = A' \qquad [C', H'] = -C' \end{cases} \tag{5.2}$$

i.e. the commutation relations of the usual oscillator by introducing

$$B' = \frac{F'(B)}{1-abF(B)}. \tag{5.3}$$

(2) $1 - ab\,F(\omega) = 0$. Let us rewrite the algebra in such an irreducible representation. We have

$$\begin{cases} [A, C] = F'(\omega)I \\ [C, H] = \dfrac{1}{a}\mathrm{e}^{-aC} \\ [A, H] = -aF'(\omega)H + \dfrac{c}{ab}\big(F''(\omega) - \mathrm{e}^{p\omega}\big)I. \end{cases} \tag{5.4}$$

If we introduce

$$A' = -H + \frac{c}{a^2bF'(\omega)}\big(F''(\omega) - \mathrm{e}^{p\omega}\big) \qquad C' = \mathrm{e}^{aC} \qquad H' = \frac{A}{aF'(\omega)} \tag{5.5}$$

from (3.7) we get

$$[A', C'] = I \qquad [A', H'] = A' \qquad [C', H'] = -C' \tag{5.6}$$

which is again identical to the usual oscillator algebra when B is the identity. However, it should be noted that the roles of A and H have to be exchanged with respect to the previous case.

Furthermore, insofar as ab is implicitly necessarily different from zero, these representations disappear in the non-quantum limit. Therefore, they have to be seen as supplementary representations ensuring that our quantum algebra is a true algebraic deformation of the initial oscillator algebra.

A similar circumstance does not happen for type II, where the action of the deformation can be seen only on the coproduct (3.14).

5.2. The fermionic case

The structure of the bialgebras defined respectively by (4.5) and (4.7), (4.8) and (4.10), (4.11) and (4.12), is investigated in greater depth.

(1). In (4.5), let us define new generators A' and C' by the formulae

$$A' = A\cosh B\sqrt{xz} + C\sqrt{\frac{x}{z}}\sinh B\sqrt{xz}$$
$$C' = A\sqrt{\frac{z}{x}}\sinh B\sqrt{xz} + C\cosh B\sqrt{xz}. \tag{5.7}$$

Then we have

$$\{A', C'\} = \frac{\sinh 2B\sqrt{xz}}{2\sqrt{xz}} \qquad A'^2 = \frac{\cosh 2B\sqrt{xz} - 1}{4z} \qquad C'^2 = \frac{\cosh 2B\sqrt{xz} - 1}{4x}$$
$$[A', H] = A' \qquad [C', H] = -C' \tag{5.8}$$

all other commutators being zero. If x, z are not simultaneously equal to zero, it can be shown from (5.8) that no element of the null square in the algebra exists that has the same commutation relations with H as A' and C'. This proves that our algebra is definitely different from the one-dimensional fermionic algebra and is a true algebraic deformation of it.

(2). If in (4.8), we introduce the new generator A' by

$$A' = A - \frac{x}{2}\frac{\mathrm{e}^{pB} - 1}{p} \tag{5.9}$$

we get the relations

$$\{A', C\} = \frac{\mathrm{e}^{pB} - 1}{p} \qquad A'^2 = -x\left(\frac{\mathrm{e}^{pB} - 1}{p}\right)^2 \qquad C^2 = 0$$
$$[A', H] = A' \qquad [C, H] = -C$$

all other commutators being zero. A similar argument as above is valid in this case and leads to the same conclusion.

(3). If in (4.11), we introduce

$$B' = \frac{\mathrm{e}^{(p+q)B} - 1}{p+q}$$

we get the following algebraic relations:

$$\{A, C\} = B' \qquad A^2 = C^2 = 0$$
$$[A, H] = A \qquad [C, H] = -C$$

all the other commutators being zero. This algebra is the non-deformed one-dimensional fermionic oscillator algebra, i.e. the graded Lie algebra $gl(1/1)$.

Nevertheless, the deformation appears in the coproduct which is now written in terms of B':

$$\begin{aligned} \Delta A &= A \otimes \left(1 + (p+q)B'\right)^{p/p+q}(-1)^H + 1 \otimes A \\ \Delta B' &= 1 \otimes B' + B' \otimes 1 + (p+q)B' \otimes B' \\ \Delta C &= C \otimes \left(1 + (p+q)B'\right)^{p/p+q}(-1)^H + 1 \otimes C \\ \Delta H &= 1 \otimes H + H \otimes 1. \end{aligned} \tag{5.10}$$

It is worthwhile noting that the previous cases provide true algebraic deformations of $gl(1/1)$.

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Appendix A. Recurrence relation for $Q_{r'}^{r,s,s'}(\eta)$ defined by (3.5)

We can prove recurrently that

$$\gamma \eta^{s'} = \eta^{s'} \gamma - a s' \eta^{s'-1}(\eta^2 - \eta). \tag{A.1}$$

Therefore the relation

$$(\gamma - sa\eta)^r \eta^{s'} = \sum_0^r Q_{r'}^{r,s,s'}(\eta) \gamma^{r'} \tag{A.2}$$

is true for $r = 1$. Now from (A.1) we derive

$$\gamma f(\eta) = f(\eta)\gamma - a(\eta^2 - \eta) f'(\eta) \tag{A.3}$$

for any sufficiently regular function $f(\eta)$. So, we can write

$$(\gamma - sa\eta)^{r+1}\eta^{s'} = \sum_0^r \left(Q_{r'}^{r,s,s'}(\eta)\gamma - a(\eta^2 - \eta)\frac{\mathrm{d}}{\mathrm{d}\eta} Q_{r'}^{r,s,s'}(\eta) - sa\eta Q_{r'}^{r,s,s'}(\eta) \right)\gamma^{r'}.$$

This proves (A.2) for any $r \in \mathbb{N}$ and gives the recurrence relation

$$Q_{r'}^{r+1,s,s'}(\eta) = Q_{r'-1}^{r,s,s'}(\eta) - \left(a(\eta^2 - \eta)\frac{\mathrm{d}}{\mathrm{d}\eta} + as\eta \right) Q_{r'}^{r,s,s'}(\eta) \tag{A.4}$$

with the solution

$$Q_{r'}^{r,s,s'}(\eta) = \binom{r}{r'}(-1)^{r-r'}\left(a(\eta^2 - \eta)\frac{\mathrm{d}}{\mathrm{d}\eta} + as\eta \right)^{r-r'} \eta^{s'}. \tag{A.5}$$

So, we immediately get

$$Q_{r'}^{r,s,s'}(1) = \binom{r}{r'}(-as)^{r-r'} \tag{A.6}$$

and in particular we have

$$Q_{r'}^{r,0,s'}(1) = \delta_{r,r'}.$$

Now, let us set

$$q^{t,s,s'} = \frac{\mathrm{d}}{\mathrm{d}\eta}\left((\eta^2-\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}+s\eta\right)^t \eta^{s'}\Big|_{\eta=1}.$$

If we remark that

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left((\eta^2-\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}+s\eta\right)^t\eta^{s'} = \left((\eta^2-\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}+s\eta\right)\frac{\mathrm{d}}{\mathrm{d}\eta}\left((\eta^2-\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}+s\eta\right)^{t-1}\eta^{s'}$$
$$+\left((2\eta-1)\frac{\mathrm{d}}{\mathrm{d}\eta}+s\right)\left((\eta^2-\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}+s\eta\right)^{t-1}\eta^{s'}$$

we get the following relation:

$$q^{t+1,s,s'} = (s+1)q^{t,s,s'} + s^{t+1} \tag{A.7}$$

the solution of which is given by

$$q^{t,s,s'} = (s+1)^t(s+s') - s^{t+1}.$$

Therefore, we finally obtain

$$\frac{\mathrm{d}Q^{r,s,s'}_{r'}}{\mathrm{d}\eta}(1) = \binom{r}{r'}(-a)^{r-r'}\left((s+1)^{r-r'}(s+s') - s^{r-r'+1}\right). \tag{A.8}$$

Appendix B. Evaluation of the $\Gamma^k_{r,s,t;r',s',t',u'}$

(1°) Type I bosonic quantum group

(a) Recurrence for $\Gamma^k_{r,s,t;r',s',t',u'}$.

As the coproduct is a homomorphism, we have

$$\Delta\beta^{k+1} = \Delta\beta^k\Delta\beta.$$

From this we get a recurrence formula for the coefficients $\Gamma^k_{r,s,t;r',s',t',u'}$, after re-ordering the monomials $\beta^{r'}\alpha^{s'}\eta^{t'}\gamma^{u'}\beta$ and $\beta^r\alpha^s\eta^t\beta$. It is done by using the following formulae deduced recurrently from the commutation relations (2.11):

$$\eta^n\beta = \beta\eta^n + na\alpha\eta^n \tag{B.1}$$

$$\alpha^n\beta = \beta\alpha^n + n\alpha^{n-1}\left(a\alpha^2 + p\alpha - b(\eta-1)\right) \tag{B.2}$$

$$\gamma^n\beta = \beta\gamma^n + \sum_0^{n-1} P^n_{n'}(\eta)\gamma^{n'}. \tag{B.3}$$

From (A.3), we obtain the following recurrence equation:

$$P^{n+1}_{n'}(\eta) = P^n_{n'-1}(\eta) - a(\eta^2-\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}P^n_{n'}(\eta)$$

the solution of which is

$$P^n_{n'}(\eta) = (-1)^{n-n'}\binom{n}{n'}ca^{n-n'-1}\left((\eta^2-\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^{n-n'-1}(\eta-1). \tag{B.4}$$

Let us set

$$P^n_{n'}(\eta) = \sum_0^{n-n'}\pi^{n,n'}_{n''}\eta^{n''}.$$

Finally we get the following recurrence formula:

$$\Gamma^{k+1}_{r,s,t;r',s',t',u'} = \sum_{u''\geqslant u'+1}\sum_{u'''=0}^{u''-u'} \pi^{u'',u'}_{u'''}\Gamma^{k}_{r,s,t;r',s',t'-u''',u''} + a(t'+s'-1)\Gamma^{k}_{r,s,t;r',s'-1,t',u'}$$
$$+p(s+s')\Gamma^{k}_{r,s,t;r',s',t',u'} - b(s'+1)\Gamma^{k}_{r,s,t;r',s'+1,t'-1,u'}$$
$$+b(s'+1)\Gamma^{k}_{r,s,t;r',s'+1,t',u'} + \Gamma^{k}_{r,s,t;r'-1,s',t',u}$$
$$+a(t+s-1)\Gamma^{k}_{r,s-1,t;r',s',t',u'} - b(s+1)\Gamma^{k}_{r,s+1,t-1;r',s',t',u'}$$
$$+b(s+1)\Gamma^{k}_{r,s+1,t;r',s',t',u'} + \Gamma^{k}_{r-1,s,t;r',s',t',u'} + \Gamma^{k}_{r,s-1,t;r',s',t',u'-1} \qquad \text{(B.5)}$$

with the convention that $\Gamma^{k}_{r,s,t;r',s',t',u'}$ with lower negative indices is zero.

(b) Formula (B.5) is the basic tool for calculating the summations involved in the expressions of various commutators of the bosonic dual algebra on the basis (3.1). We give some significant examples which appear in $[A, C]$ and $[A, H]$ for $x = -a$.

(i) $\sum_{t,t',u'}\Gamma^{k}_{0,0,t;0,0,t',u'}x^{u'}$. From (B.5), we get

$$\sum_{t,t',u'}\Gamma^{k+1}_{0,0,t;0,0,t',u'}x^{u'} = \sum_{t,t',u'}\sum_{u''\geqslant u'+1}\sum_{u'''=0}^{u''-u'}\pi^{u''u'}_{u'''}\Gamma^{k}_{0,0,t;0,0,t'-u''',u''}x^{u'}$$
$$= \sum_{u'}\sum_{u''\geqslant u'+1}\left(\sum_{u'''=0}^{u''-u'}\pi^{u''u'}_{u'''}\right)x^{u'}\sum_{t,t'}\Gamma^{k}_{0,0,t;0,0,t',u''}.$$

But, according to (B.4), we have

$$\sum_{u'''=0}^{u''-u'}\pi^{u'',u'}_{u'''} = P^{u''}_{u'}(1) = 0. \qquad \text{(B.6)}$$

Obviously as

$$\sum_{t,t',u'}\Gamma^{0}_{0,0,t;0,0,t',u'}x^{u'} = 1$$

we finally get

$$\sum_{t,t',u'}\Gamma^{k}_{0,0,t;0,0,t',u'}x^{u'} = \delta_{k0}. \qquad \text{(B.7)}$$

(ii) $\sum_{t,t',u'} t\Gamma^{k}_{0,0,t;0,0,t',u'}x^{u'}$. From equations (B.5) and (B.6), we obtain

$$\sum_{t,t',u'} t\Gamma^{k+1}_{0,0,t;0,0,t',u'}x^{u'} = -b\sum_{t,t',u'}\Gamma^{k}_{0,1,t;0,0,t',u'}x^{u'}. \qquad \text{(B.8)}$$

But, from equation (B.5), we also have

$$\sum_{t,t',u'}\Gamma^{k+1}_{0,1,t;0,0,t',u'}x^{u'} = p\sum_{t,t',u'}\Gamma^{k}_{0,1,t;0,0,t',u'}x^{u'} + a\sum_{t,t',u'} t\Gamma^{k}_{0,0,t;0,0,t',u'}x^{u'}$$
$$+x\sum_{t,t',u'}\Gamma^{k}_{0,0,t;0,0,t',u'}x^{u'}.$$

Using equations (B.7) and (B.8), we get

$$\sum_{t,t',u'}\Gamma^{k+1}_{0,1,t;0,0,t',u'}x^{u'} = p\sum_{t,t',u'}\Gamma^{k}_{0,1,t;0,0,t',u'}x^{u'} - ab\sum_{t,t',u'}\Gamma^{k-1}_{0,1,t;0,0,t',u'}x^{u'} + x\delta_{k0}.$$

So, we can write

$$\sum_{t,t',u'} \Gamma^k_{0,1,t;0,0,t',u'} x^{u'} = x R^{k-1}(p)(1-\delta_{k0}) \tag{B.9}$$

where the polynomials $R^k(p)$ verify

$$\begin{cases} R^0(p) = 1 \qquad R^1(p) = p \\ R^{k+1}(p) = pR^k(p) - abR^{k-1}(p) \qquad k = 1, 2, \ldots \end{cases} \tag{B.10}$$

and finally, we obtain

$$\sum_{t,t',u'} t\Gamma^k_{0,0,t;0,0,t',u'} x^{u'} = -bx R^{k-2}(p)(1-\delta_{k0}-\delta_{k1}). \tag{B.11}$$

To solve equation (B.10), we introduce

$$R(y) = \sum_0^\infty \frac{y^k}{k!} R^k(p).$$

The recurrence relation gives the differential equation:

$$R''(y) - pR'(y) + abR(y) = 0$$

with the solution

$$R(y) = \frac{\rho_1 \mathrm{e}^{\rho_1 y} - \rho_2 \mathrm{e}^{\rho_2 y}}{\rho_1 - \rho_2} = F''(y).$$

where ρ_1, ρ_2 are given by (3.9).

(iii) $\sum_{t,t',u'} \Gamma^k_{0,0,t;0,0,t',u'} t' x^{u'}$. From (B.5), we get

$$\sum_{t,t',u'} \Gamma^{k+1}_{0,0,t;0,0,t',u'} t' x^{u'} = \sum_{u'} \sum_{u''\geqslant u'+1} \left(\sum_{u'''=0}^{u''-u'} \pi^{u''u'}_{u'''} u''' \right) x^{u'} \sum_{t,t'} \Gamma^k_{0,0,t;0,0,t',u'} x^{u'}$$

$$-b \sum_{t,t',u'} \Gamma^k_{0,0,t;0,1,t',u'} x^{u'}. \tag{B.12}$$

But, according to equation (B.4), we have

$$\sum_{u'''=0}^{u''-u'} \pi^{u''u'}_{u'''} u''' = \frac{\mathrm{d}}{\mathrm{d}\eta} P^{u''}_{u'}(\eta) \bigg|_{\eta=1} = (-1)^{u''-u'} \binom{u''}{u'} c a^{u''-u'-1}. \tag{B.13}$$

So, the first term in the right-hand side of (B.12) reads:

$$\frac{c}{a} \sum_{t,t',u''} \left((x-a)^{u''} - x^{u''} \right) \Gamma^k_{0,0,t;0,0,t',u''} = 0$$

according to (B.7).

Now, from (B.5), we have

$$\sum_{t,t',u'} \Gamma^{k+1}_{0,0,t;0,1,t',u'} x^{u'} = a \sum_{t,t',u'} \Gamma^k_{0,0,t;0,0,t',u'} t' x^{u'} + p \sum_{t,t',u'} \Gamma^k_{0,0,t;0,1,t',u'} x^{u'}$$

$$= p \sum_{t,t',u'} \Gamma^k_{0,0,t;0,1,t',u'} x^{u'} - ab \sum_{t,t',u'} \Gamma^{k-1}_{0,0,t;0,1,t',u'} x^{u'}.$$

This implies

$$\sum \Gamma^k_{0,0,t;0,1,t',u'} x^{u'} = 0 \tag{B.14}$$

since this is true for $k = 0, 1$. Therefore, we have

$$\sum_{t,t',u'} \Gamma^k_{0,0,t;0,0,t',u'} t' x^{u'} = 0. \tag{B.15}$$

(iv) $\sum_{t,t'} (\Gamma^k_{0,1,t;0,0,t',0} t' - t\Gamma^k_{0,0,t;0,1,t',0})$. From equations (B.5), (B.9) and (B.13) we get

$$\sum_{t,t'} (\Gamma^{k+1}_{0,1,t;0,0,t',0} t' - t\Gamma^{k+1}_{0,0,t;0,1,t',0})$$

$$= p \sum_{t,t'} (\Gamma^k_{0,1,t;0,0,t',0} t' - t\Gamma^k_{0,0,t;0,1,t',0}) - cR^{k-1}(p)(1 - \delta_{k0}). \tag{B.16}$$

Let us set

$$G(y) = \sum_0^\infty \frac{y^k}{k!} \sum_{t,t'} (\Gamma^k_{0,1,t;0,0,t',0} t' - t\Gamma^k_{0,0,t;0,1,t',0}) \qquad G(0) = 0.$$

From (B.16), we deduce the differential equation:

$$G'(y) = pG(y) - cF'(y)$$

with the solution:

$$G(y) = \frac{c}{ab}\big(F''(y) - \mathrm{e}^{py}\big)$$

from which the final result can easily be written.

(2°) Type II bosonic quantum group

The recurrence relation for $\Gamma^k_{r,s,t;r',s',t',u'}$ is easier to obtain. Indeed, after we recurrently prove the following equations:

$$\gamma^n \beta = \beta\gamma^n + q\gamma^n(1 - \delta_{n0}) + nc(1 - \eta)\gamma^{n-1}$$

$$\alpha^n \beta = \beta\alpha^n + p\alpha^n(1 - \delta_{n0}) + nb(1 - \eta)\alpha^{n-1}$$

we get

$$\begin{aligned}\Gamma^{k+1}_{r,s,t;r',s',t',u'} = {} & \big(p(2 - \delta_{s0} - \delta_{s'0}) + q(1 - \delta_{u'0})\big)\Gamma^k_{r,s,t;r',s',t',u'} + b(s'+1)\Gamma^k_{r,s,t;r',s'+1,t',u'} \\ & - b(s'+1)\Gamma^k_{r,s,t;r',s'+1,t'-1,u'} + b(s+1)\Gamma^k_{r,s+1,t;r',s',t',u'} \\ & - b(s+1)\Gamma^k_{r,s+1,t-1;r',s',t',u'} + c(u'+1)\Gamma^k_{r,s,t;u',r',s',t',u'+1} \\ & - c(u'+1)\Gamma^k_{r,s,t;r',s',t'-1,u'+1} + \Gamma^k_{r,s,t;r'-1,s',t',u'} + \Gamma^k_{r-1,s,t;r',s',t',u'} \\ & + \Gamma^k_{r,s-1,t;r',s',t',u'-1}\end{aligned} \tag{B.17}$$

with the same convention as in (B.5).

We proceed as above to evaluate the various needed summations in taking (3.11) into account as we perform the limit $a = 0$. We illustrate the method by a unique example concerning $[C, H]$.

Let us consider $\sum_{t,t'} t\Gamma^k_{0,0,t;0,0,t',1}$. From equation (B.17), we have

$$\sum_{t,t'} t\Gamma^{k+1}_{0,0,t;0,0,t',1} = q \sum_{t,t'} t\Gamma^k_{0,0,t;0,0,t',1} - b \sum_{t,t'} \Gamma^k_{0,1,t;0,0,t',1}. \tag{B.18}$$

But, again from (B.17), we successively get

$$\sum_{t,t'} \Gamma^{k+1}_{0,1,t;0,0,t',1} = (p+q) \sum_{t,t'} \Gamma^{k}_{0,1,t;0,0,t',1} + \sum_{t,t'} \Gamma^{k}_{0,0,t;0,0,t',0}$$

$$\sum_{t,t'} \Gamma^{k+1}_{0,0,t;0,0,t',0} = 0 \qquad k = 0, 1, \ldots$$

This implies

$$\sum_{t,t'} \Gamma^{k}_{0,0,t;0,0,t',0} = \delta_{k0} \qquad \sum_{t,t'} \Gamma^{k}_{0,1,t;0,0,t',1} = (p+q)^{k-1}(1-\delta_{k0}).$$

Let us set

$$g(x) = \sum \frac{x^k}{k!} \sum_{t,t'} t\Gamma^{k}_{0,0,t;0,0,t',1}.$$

From equation (B.18), we obtain the differential equation

$$g'(x) = qg(x) - b\frac{\mathrm{e}^{(p+q)x} - 1}{p+q} \qquad g(0) = 0.$$

The solution is

$$g(x) = -\frac{b}{q}\frac{\mathrm{e}^{(p+q)x} - 1}{p+q} - \frac{b}{pq}(1 - \mathrm{e}^{qx})$$

from which the final result is easily written.

Appendix C. Proof of useful lemmas

First we consider type I bosonic quantum groups.

Lemma C.1. For $w \in \mathbb{N}$, we have

$$(B^w, \beta^k\alpha^\ell\eta^m\gamma^n) = w!\delta_{kw}\delta_{\ell 0}\delta_{n0}. \tag{C.1}$$

Proof . Equation (C.1) is true for $w = 1$ according to the definition of B. Let us assume (C.1) is verified up to some w. Since we have

$$(B^{w+1}, \beta^k\alpha^\ell\eta^m\gamma^n) = (B^w \otimes B, \Delta\beta^k\alpha^\ell\eta^m\gamma^n)$$

according to equations (3.6) and (A.6) we get

$$(B^{w+1}, \beta^k\alpha^\ell\eta^m\gamma^n) = w! \sum_{t,t'} \Gamma^{k}_{w,0,t;1,0,t',0}\delta_{\ell 0}\delta_{n0}.$$

From equation (B.5), we can write

$$\sum_{t,t'} \Gamma^{k+1}_{w,0,t;1,0,t',0} = \sum_{t,t'} \Gamma^{k}_{w,0,t;0,0,t',0} + \sum_{t,t'} \Gamma^{k}_{w-1,0,t;1,0,t',0}. \tag{C.2}$$

But, according to equations (B.5) and (B.7), we have

$$\sum_{t,t'} \Gamma^{k+1}_{w,0,t;0,0,t',0} = \sum_{t,t'} \Gamma^{k}_{w-1,0,t;0,0,t',0} = \sum_{t,t'} \Gamma^{k+1-w}_{0,0,t;0,0,t',0} = \delta_{k+1,w}.$$

Substituting this in (C.2) and solving the recurrence, we get:

$$\sum_{t,t'} \Gamma^{k}_{w,0,t;1,0,t',0} = (w+1)\delta_{k,w}$$

which completes the proof. □

Lemma C.2. For $w \in \mathbb{N}$, we have

$$(C^w, \beta^k \alpha^\ell \eta^m \gamma^n) = w! \delta_{k0} \delta_{\ell 0} \delta_{nw}.$$

Proof . We proceed recurrently as above. First, we have

$$(C^{w+1}, \beta^k \alpha^\ell \eta^m \gamma^n) = w! \sum_{t,t'} \Gamma^k_{0,0,t;0,0,t',1} \delta_{\ell 0} \delta_{nw} + w! \sum_{t,t'} \Gamma^k_{0,0,t;0,0,t',0} \delta_{\ell 0} \sum_u \Delta^n_{uw,1}.$$

According to (B.7), the first term in the right-hand side is zero. Moreover, it can easily be proved that

$$\sum_u \Delta^n_{uw,1} = (w+1) \delta_{n,w+1}$$

so that using (B.7) again, we get

$$(C^{w+1}, \beta^k \alpha^\ell \eta^m \gamma^n) = (w+1)! \delta_{k0} \delta_{\ell 0} \delta_{n,w+1}$$

which completes the proof. □

Lemma C.3. For $w \in \mathbb{N}$, we have

$$(CB^w, \beta^k \alpha^\ell \eta^m \gamma^n) = w! \delta_{kw} \delta_{\ell 0} \delta_{n1}.$$

Proof . By definition

$$\begin{aligned}(CB^w, \beta^k \alpha^\ell \eta^m \gamma^n) &= (C \otimes B^w, \Delta \beta^k \alpha^\ell \eta^m \gamma^n) \\ &= w! \sum_{t,t'} \Gamma^k_{0,0,t;w,0,t',0} \delta_{\ell 0} \delta_{n1}.\end{aligned}$$

Using equations (B.5) and (B.7), we have

$$\sum_{t,t'} \Gamma^{k+1}_{0,0,t;w,0,t',0} = \sum_{t,t'} \Gamma^k_{0,0,t;w-1,0,t',0} = \sum_{t,t'} \Gamma^{k+1-w}_{0,0,t;0,0,t',0} = \delta_{k+1,w}$$

which completes the proof. □

The same results are true for the type II bosonic quantum group and can be similarly proved by using (B.17) instead of (B.5). We leave the proof to the reader.

Appendix D. Recurrence relations and some examples of evaluation

By writing $\Delta \beta^{k+1} = \Delta \beta^k \Delta \beta$ and re-ordering according to the algebraic relations (2.17) with $p = q = 0$ (i.e. the type I fermionic quantum group), we get

$$\begin{aligned}\Gamma^{k+1,ij}_{rs;r's'} &= \Gamma^{k,ij}_{rs;r'-1,s'} + \Gamma^{k,ij}_{r-1,s;r's'} + ij \Delta^{k,i+1,j}_{rs;r's'} + \big(xi(j+1) + zj(i+1)\big) \\ &\quad \times \Big(\tfrac{1}{2}(\Delta^{k,i+1,j}_{rs;r's'} - \Delta^{k,i+1,j}_{r,s-2j;r',s'-2i}) - \Delta^{k,ij}_{rs;r',s'-1} - \Delta^{k,i+1,j+1}_{r,s-1;r's'}\Big) \\ &\quad + (i+1)(j+1) \tfrac{1}{4} xz (\Delta^{k,i+1,j}_{rs;r's'} - \Delta^{k,i+1,j}_{r,s-2;r's'} \Delta^{k,i+1,j}_{rs;r',s'-2} + \Delta^{k,i+1,j}_{r,s-2;r',s'-2}).\end{aligned} \tag{D.1}$$

$$\begin{aligned}\Delta^{k+1,ij}_{rs;r's'} &= \Delta^{k,ij}_{rs;r'-1,s'} + \Delta^{k,ij}_{r-1,s;r's'} + i(j+1) \Gamma^{k,i+1,j}_{rs;r's'} \\ &\quad + ij \tfrac{1}{2} x \big(\Gamma^{k,i+1,j}_{rs;r's'} - \Gamma^{k,i+1,j}_{rs;r',s'-2}\big) + (i+1)(j+1) \tfrac{1}{2} z \big(\Gamma^{k,i+1,j}_{rs;r's'} - \Gamma^{k,i+1,j}_{r,s-2;r's'}\big) \\ &\quad - \big((x(i+1)j + zi(j+1)\big) \Gamma^{k,ij}_{rs;r',s'-1} - \big(xi(j+1) + z(i+1)j\big) \Gamma^{k,i+1,j+1}_{r,s-1;r's'} \\ &\quad + (i+1) j \tfrac{1}{4} xz \big(\Gamma^{k,i+1,j}_{rs;r's'} - \Gamma^{k,i+1,j}_{rs;r',s'-2} - \Gamma^{k,i+1,j}_{r,s-2;r's'} + \Gamma^{k,i+1,j}_{r,s-2;r',s'-2}\big).\end{aligned} \tag{D.2}$$

By convention, the coefficients with negative lower indices are equal to zero, and $i+1$, $j+1$ are taken mod 2.

We now give some examples of evaluation:

(i) $\sum_{s,s'} \Gamma^{k,00}_{0s;0s'}$. From the recurrence, we readily get

$$\sum_{s,s'} \Gamma^{k+1,00}_{0s;0s'} = 0 \qquad k = 0, 1, \ldots$$

so that we have

$$\sum_{s,s'} \Gamma^{k,00}_{0s;0s'} = \delta_{k0}.$$

(ii) From this result, we deduce

$$\sum_{s,s'} \Gamma^{k,00}_{1s;0s'} = \sum_{s,s'} \Gamma^{k,00}_{0s;1s'} = \delta_{k1}$$

since the recurrence gives

$$\sum_{s,s'} \Gamma^{k+1,00}_{1s;0s'} = \sum_{s,s'} \Gamma^{k,00}_{0s;0s'} = \sum_{s,s'} \Gamma^{k+1,00}_{0s;1s'}.$$

(iii) A little more computation is needed to evaluate the summations involved in the anticommutator $\{A, C\}$. Indeed, we have successively

$$\begin{aligned}
&\sum_{s,s'} \Delta^{k+1,10}_{0s;0s'} = \delta_{k0} - z \sum_{s,s'} \Gamma^{k,10}_{0s,0s'} - x \sum_{s,s'} \Gamma^{k,01}_{0s;0s'} \\
&\sum_{s,s'} \Gamma^{k+1,10}_{0s;0s'} = -x \sum_{s,s'} \Gamma^{k,10}_{0s,0s'} - x \sum_{s,s'} \Delta^{k,01}_{0s;0s'} \\
&\sum_{s,s'} \Delta^{k+1,01}_{0s;0s'} = -z \sum_{s,s'} \Gamma^{k,10}_{0s,0s'} - x \sum_{s,s'} \Delta^{k,01}_{0s;0s'}.
\end{aligned} \tag{D.3}$$

Let us define

$$X_k = -z \sum_{s,s'} \Gamma^{k,10}_{0s,0s'} - x \sum \Gamma^{k,01}_{0s,0s'}.$$

We have the recurrence relation

$$X_{k+1} = 2xz\delta_{k1} + 4xzX_{k-1}$$

with the initial values

$$X_0 = X_1 = 1$$

the solution of which is given by

$$X_k = \tfrac{1}{4}(4xz)^{k/2}\big(1 + (-1)^k\big) - \tfrac{1}{2}\delta_{k0}.$$

From this result, we readily deduce the values of

$$\sum_{s,s'} \Delta^{k,01}_{0s;0s'} \qquad \sum_{s,s'} \Gamma^{k,10}_{0s;0s'} \qquad \sum_{s,s'} \Gamma^{k,01}_{0s;0s'} \qquad \text{and} \qquad \sum_{s,s'} \Delta^{k,10}_{0s;0s'}.$$

The other evaluations are left to the reader.

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